Review of Lecture 3

- Linear models use the '**signal**':

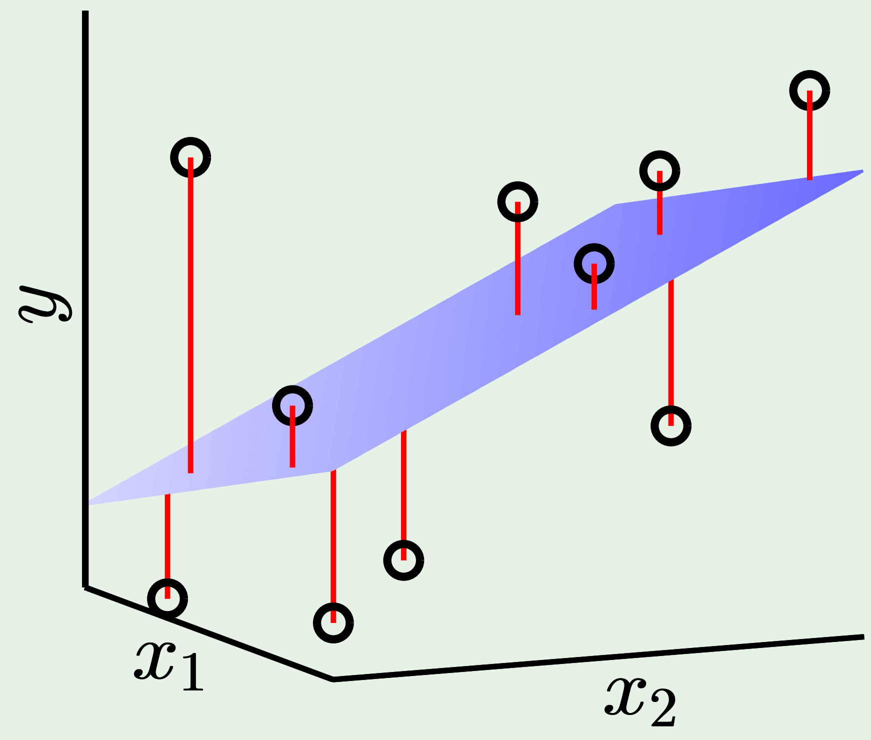
$$\sum_{i=0}^{d} w_i x_i = \mathbf{w}^\intercal \mathbf{x}$$

  - Classification: $h(\mathbf{x}) = \mathrm{sign}(\mathbf{w}^\intercal \mathbf{x})$
  - Regression: $h(\mathbf{x}) = \mathbf{w}^\intercal \mathbf{x}$

- Linear regression algorithm:

$$\mathbf{w} = (\mathrm{X}^\intercal \mathrm{X})^{-1} \mathrm{X}^\intercal \mathbf{y}$$

"one-step learning"



- Nonlinear transformation:
  - $\mathbf{w}^\intercal \mathbf{x}$ is linear in $\mathbf{w}$
  - Any $\mathbf{x} \xrightarrow{\Phi} \mathbf{z}$ preserves this linearity.
  - Example: $(x_1, x_2) \xrightarrow{\Phi} (x_1^2, x_2^2)$

Learning From Data

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Lecture 4: Error and Noise

Sponsored by Caltech's Provost Office, E&AS Division, and IST • Thursday, April 12, 2012

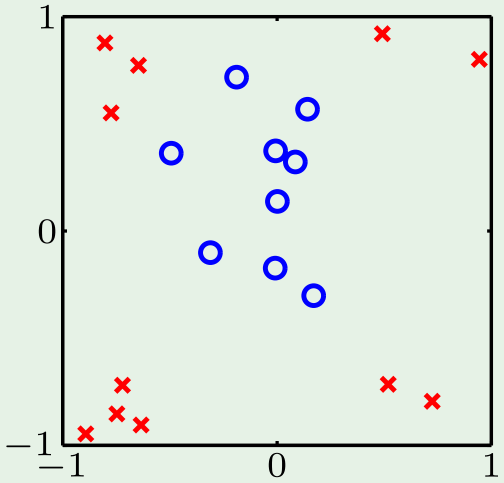
Outline

- Nonlinear transformation (continued)

- Error measures

- Noisy targets

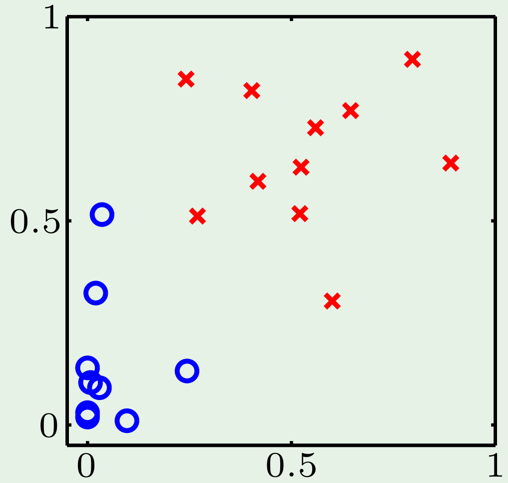
- Preamble to the theory



1. Original data

$\mathbf{x}_n \in \mathcal{X}$

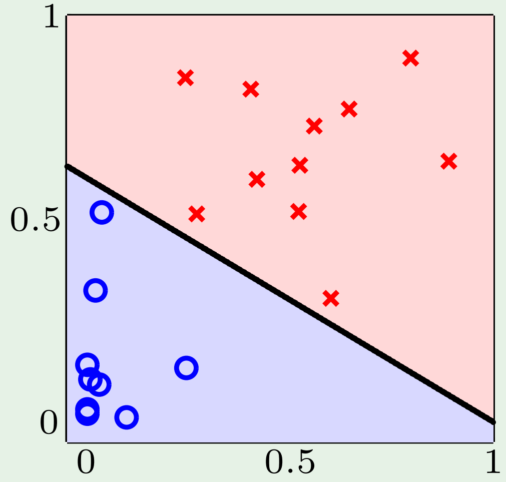
$\xrightarrow{\ \Phi\ }$



2. Transform the data

$\mathbf{z}_n = \Phi(\mathbf{x}_n) \in \mathcal{Z}$

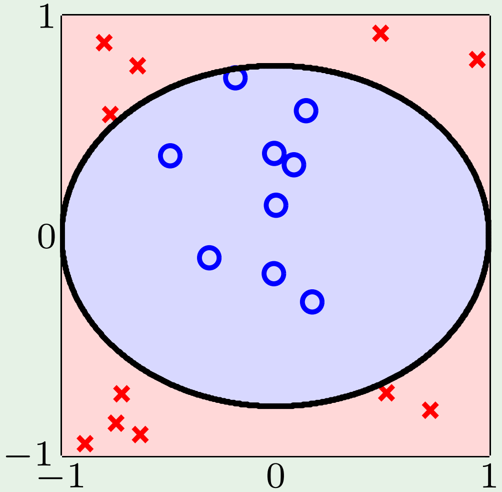
$\downarrow$



3. Separate data in $\mathcal{Z}$-space

$\tilde{g}(\mathbf{z}) = \text{sign}(\tilde{\mathbf{w}}^{\mathsf{T}}\mathbf{z})$

$\xleftarrow{\ `\Phi^{-1}\text{'}\ }$



4. Classify in $\mathcal{X}$-space

$g(\mathbf{x}) = \tilde{g}(\Phi(\mathbf{x})) = \text{sign}(\tilde{\mathbf{w}}^{\mathsf{T}}\Phi(\mathbf{x}))$

What transforms to what

$$\mathbf{x} = (x_0, x_1, \cdots, x_d) \quad \xrightarrow{\Phi} \quad \mathbf{z} = (z_0, z_1, \cdots\cdots\cdots, z_{\tilde{d}})$$

$$\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N \quad \xrightarrow{\Phi} \quad \mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_N$$

$$y_1, y_2, \cdots, y_N \quad \xrightarrow{\Phi} \quad y_1, y_2, \cdots, y_N$$

No weights in $\mathcal{X}$ $\qquad$ $\tilde{\mathbf{w}} = (w_0, w_1, \cdots\cdots\cdots, w_{\tilde{d}})$

$$g(\mathbf{x}) \quad = \quad \mathrm{sign}(\tilde{\mathbf{w}}^{\mathsf{T}} \Phi(\mathbf{x}))$$

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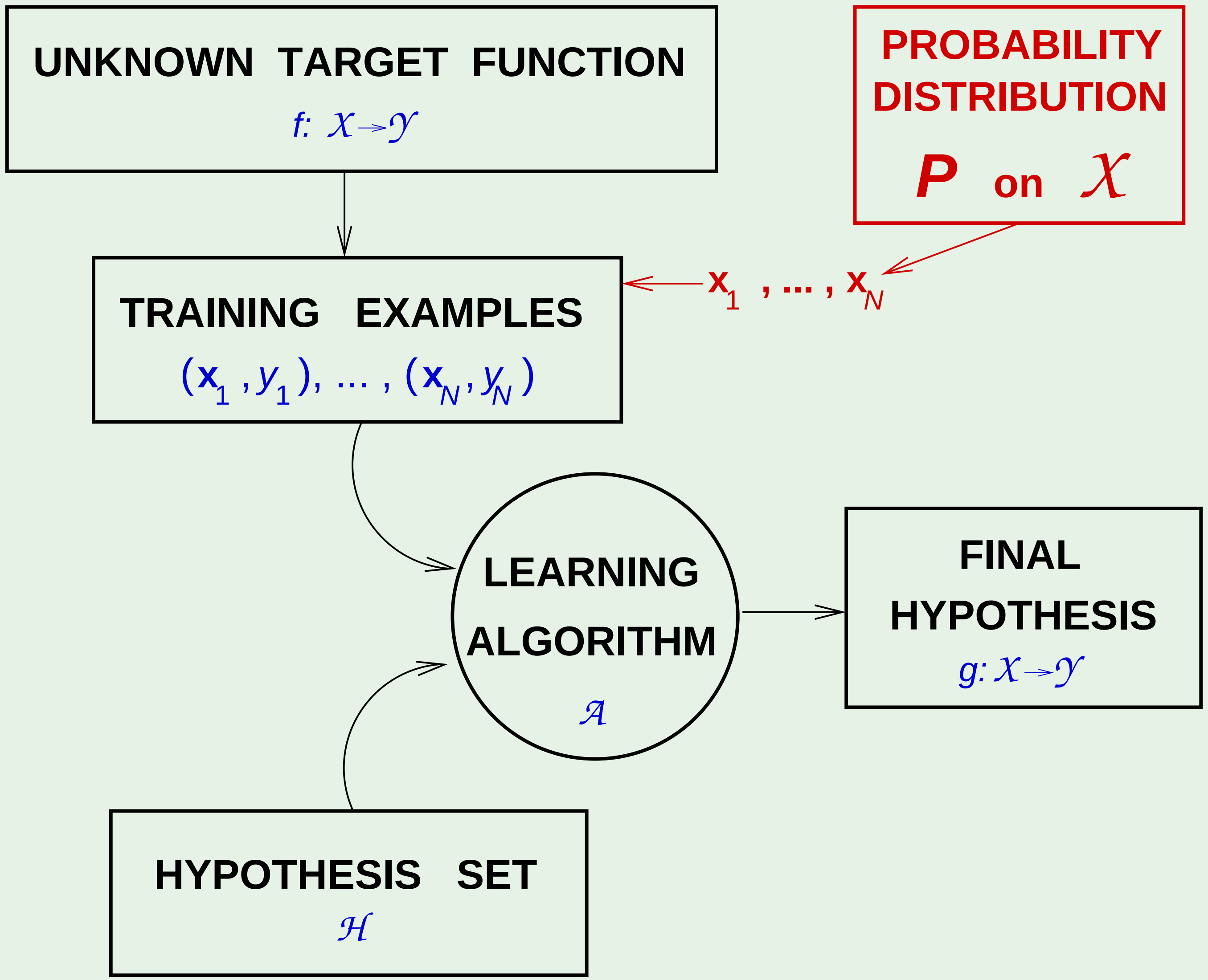
The learning diagram - where we left it



Error measures

What does "$h \approx f$" mean?

Error measure: $E(h, f)$

Almost always *pointwise definition*: $\mathrm{e}\left(h(\mathbf{x}), f(\mathbf{x})\right)$

Examples:

Squared error: $\mathrm{e}\left(h(\mathbf{x}), f(\mathbf{x})\right) = \left(h(\mathbf{x}) - f(\mathbf{x})\right)^2$

Binary error: $\mathrm{e}\left(h(\mathbf{x}), f(\mathbf{x})\right) = [\![h(\mathbf{x}) \neq f(\mathbf{x})]\!]$

From pointwise to overall

Overall error $E(h, f)$ = average of pointwise errors $\mathrm{e}\left(h(\mathbf{x}), f(\mathbf{x})\right)$.

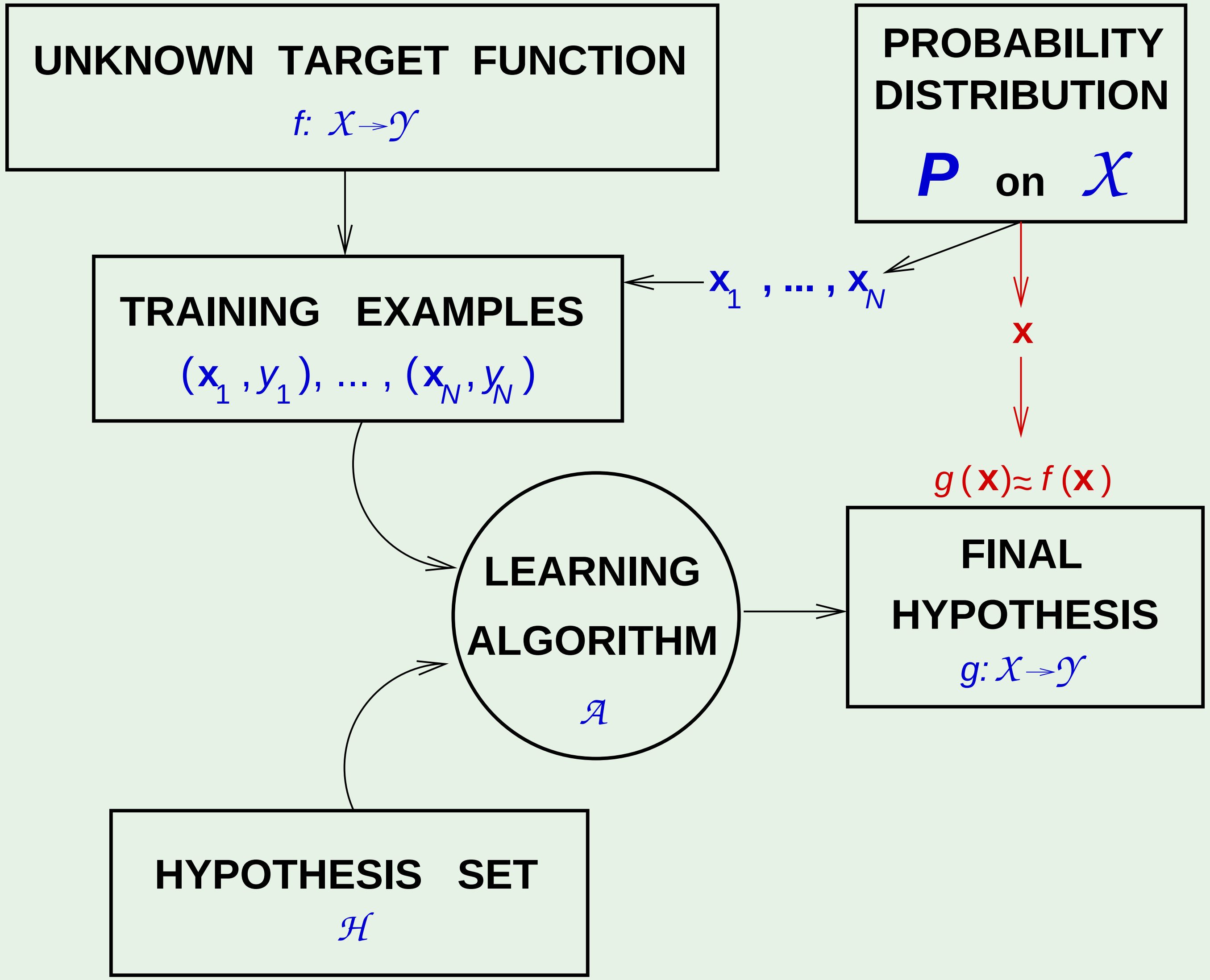
In-sample error:

$$E_{\text{in}}(h) = \frac{1}{N} \sum_{n=1}^{N} \mathrm{e}\left(h(\mathbf{x}_n), f(\mathbf{x}_n)\right)$$

Out-of-sample error:

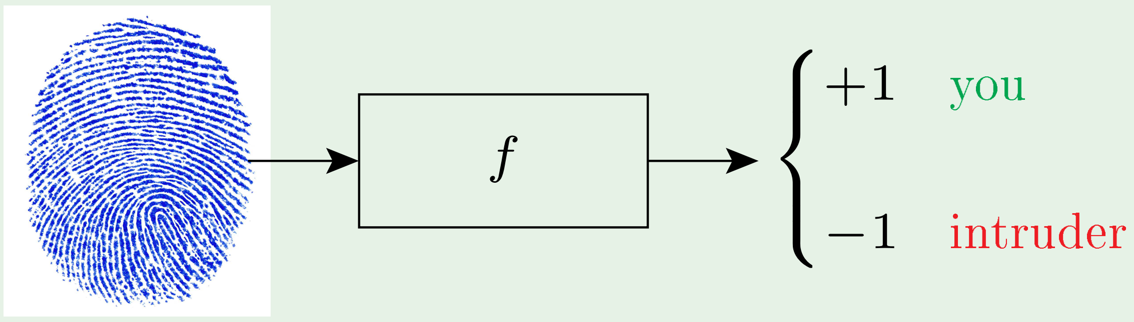
$$E_{\text{out}}(h) = \mathbb{E}_{\mathbf{x}}\left[\mathrm{e}\left(h(\mathbf{x}), f(\mathbf{x})\right)\right]$$

The learning diagram - with pointwise error



How to choose the error measure

Fingerprint verification:



Two types of error:

false accept and *false reject*

How do we penalize each type?

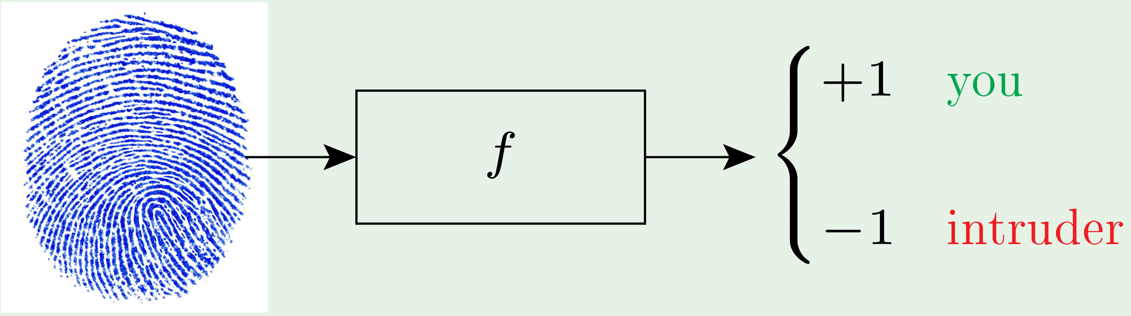
		f	
		$+1$	-1
h	$+1$	no error	*false accept*
	-1	*false reject*	no error

The error measure - for supermarkets

Supermarket verifies fingerprint for discounts

False reject is costly; customer gets annoyed!

False accept is minor; gave away a discount
and intruder left their fingerprint ☺



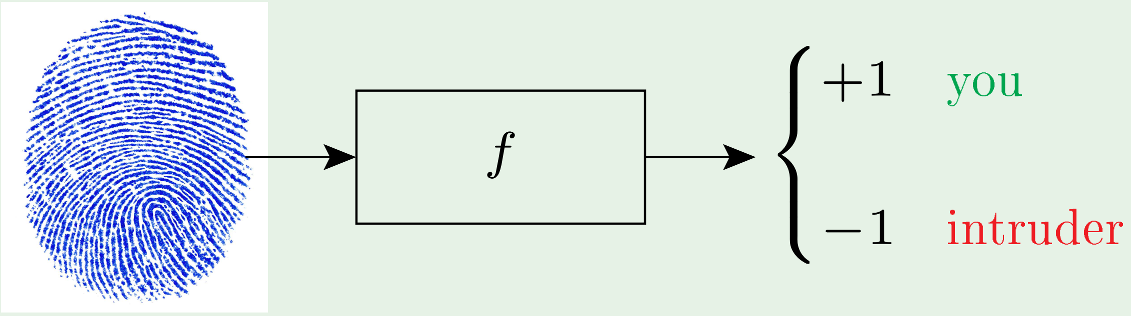
		f: $+1$	f: -1
h	$+1$	0	1
	-1	10	0

The error measure - for the CIA

CIA verifies fingerprint for security

False accept is a disaster!

False reject can be tolerated
Try again; you are an employee ☺



		f: $+1$	f: -1
h	$+1$	0	1000
	-1	1	0

Take-home lesson

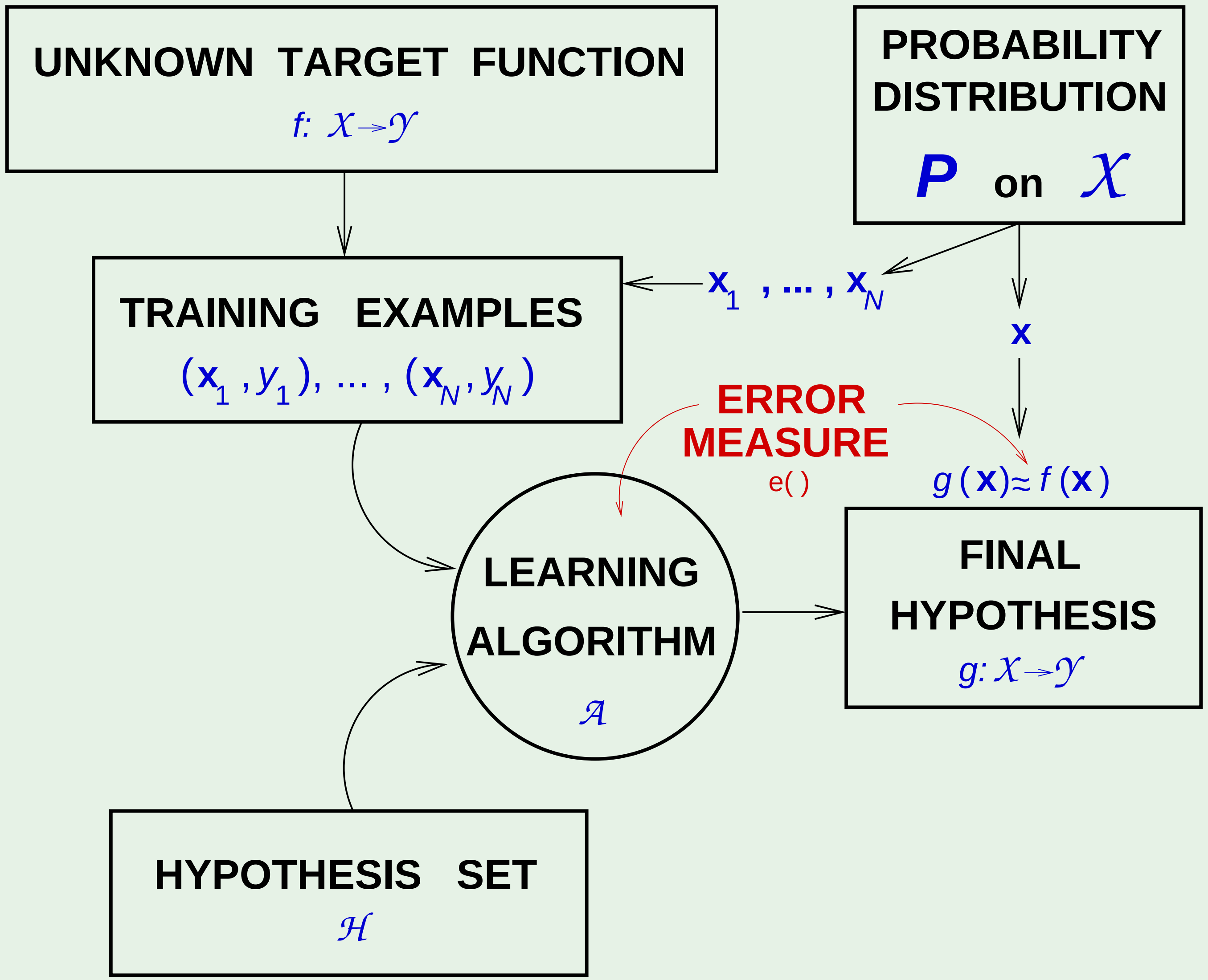
The error measure should be specified by the user.

Not always possible. Alternatives:

Plausible measures: squared error $\equiv$ Gaussian noise

Friendly measures: closed-form solution, convex optimization

The learning diagram - with error measure



Noisy targets

The 'target function' is not always a *function*

Consider the credit-card approval:

age	23 years
annual salary	\$30,000
years in residence	1 year
years in job	1 year
current debt	\$15,000
···	···

two 'identical' customers $\longrightarrow$ two different behaviors

Target 'distribution'

Instead of $y = f(\mathbf{x})$, we use target *distribution*:

$$P(y \mid \mathbf{x})$$

$(\mathbf{x}, y)$ is now generated by the joint distribution:

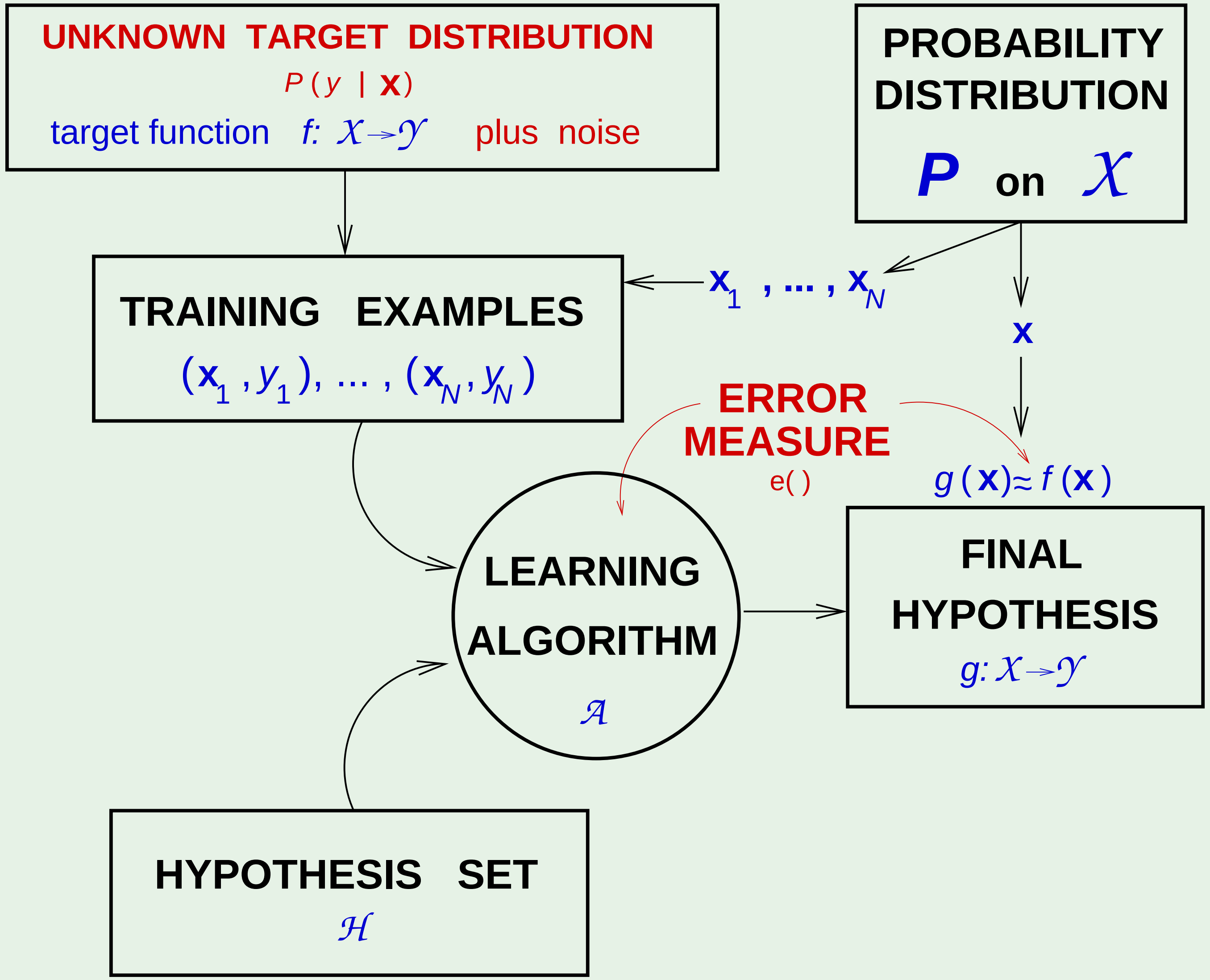
$$P(\mathbf{x})P(y \mid \mathbf{x})$$

Noisy target = deterministic target $f(\mathbf{x}) = \mathbb{E}(y|\mathbf{x})$ plus noise $y - f(\mathbf{x})$

Deterministic target is a special case of noisy target:

$$P(y \mid \mathbf{x}) \text{ is zero except for } y = f(\mathbf{x})$$

The learning diagram - including noisy target



Distinction between $P(y|\mathbf{x})$ and $P(\mathbf{x})$

Both convey probabilistic aspects of $\mathbf{x}$ and y

The target distribution $P(y \mid \mathbf{x})$

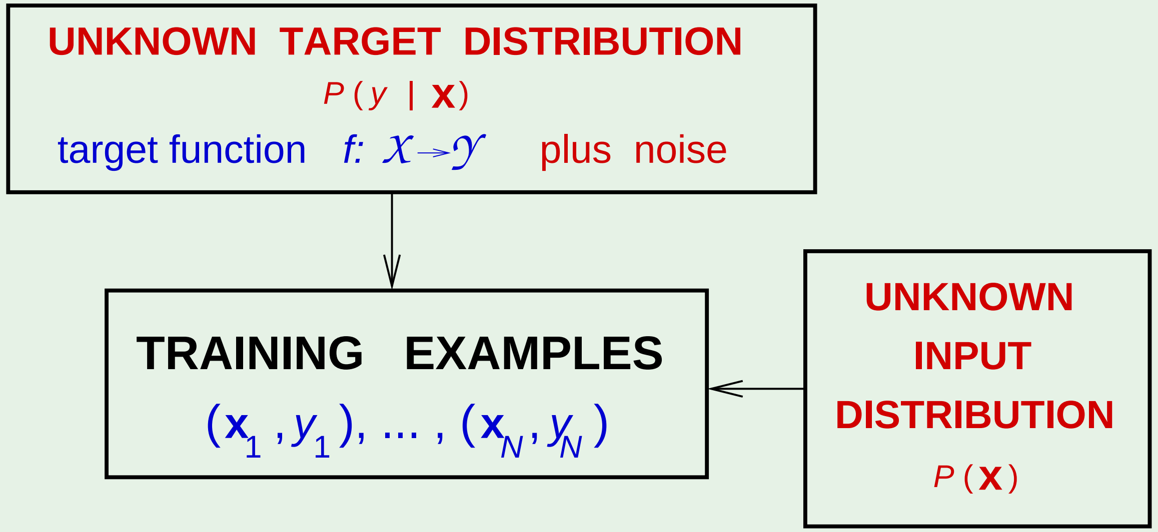
is what we are trying to learn

The input distribution $P(\mathbf{x})$

quantifies relative importance of $\mathbf{x}$

Merging $P(\mathbf{x})P(y|\mathbf{x})$ as $P(\mathbf{x}, y)$

mixes the two concepts



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What we know so far

Learning is feasible. It is likely that

$$E_{\text{out}}(g) \approx E_{\text{in}}(g)$$

Is this learning?

We need $g \approx f$, which means

$$E_{\text{out}}(g) \approx 0$$

The 2 questions of learning

$E_{\text{out}}(g) \approx 0$ is achieved through:

$$\underbrace{E_{\text{out}}(g) \approx E_{\text{in}}(g)}_{\text{Lecture 2}} \qquad \text{and} \qquad \underbrace{E_{\text{in}}(g) \approx 0}_{\text{Lecture 3}}$$

Learning is thus split into 2 questions:

1. Can we make sure that $E_{\text{out}}(g)$ is close enough to $E_{\text{in}}(g)$?
2. Can we make $E_{\text{in}}(g)$ small enough?

What the theory will achieve

Characterizing the feasibility of learning for infinite M

Characterizing the tradeoff:

Model complexity	$\uparrow$	E_{in}	$\downarrow$
Model complexity	$\uparrow$	$E_{\text{out}} - E_{\text{in}}$	$\uparrow$